THE PROBABILITY DISTRIBUTION OF THE MEASURE OF A RANDOM LINEAR SET

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "THE PROBABILITY DISTRIBUTION OF THE MEASURE OF A RANDOM LINEAR SET", submitted by DENNIS D. BECK in partial fulfillment of the requirements for the degree of Master of Science.

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ABSTRACT

Various authors have carried out studies in the general area of the measure of a random set. W.L. Stevens considered arcs of length D $(0 < D < 1)$ at random points on the circumference of a circle of length unity and derived the probability that the circumference is completely covered. A somewhat similar but non-circular problem was dealt with by D.F. Votaw Jr. He derived the probability function of the measure of a random linear set. C. Domb also considered a problem of random intervals on the line. H.E. Robbins derived general formulae for the moments of the measure of any random set. The results of these authors and others are reviewed in the first two chapters of this thesis.

The third chapter is devoted to the derivation of the probability function of the portion of the circumference of a circle of unit length which is covered by arcs, of length D $(0 < D < 1)$, at random points on the circumference. The first two moments of the coverage are also derived.

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CHAPTER I

INTRODUCTION

Consider $n+1$ points $z_i (i=0,1,\ldots,n)$ selected independently and at random from the interval $(0,1)$, the distribution of any z_i being the uniform distribution with distribution function

$$(1.1) \qquad F(z) = z \qquad (0 \leq z \leq 1) .$$

Let the $n+1$ points be ordered in ascending order of magnitude as $x_i (i=0,1,\ldots n)$. The totality of all cases in which two of the x_i are equal has zero probability, and such cases can be excluded without affecting the problem. Associate with each x_i an interval I where

$$(1.2) \quad I = \begin{cases} (x_i, x_i+D) & \text{if} \quad x_i+D \leq 1 \; (0 < D < 1; i=0,1,\ldots,n), \\ (x_i,1) \cup (0,x_i+D-1) & \text{if} \quad x_i+D > 1 \; . \end{cases}$$

Let X denote the random set which is the point set sum of the $n+1$ intervals $\{I\}$ and let $\mu(X)$ be its measure. The range of $\mu(X)$ is $D \leq \mu(X) \leq m$, where m denotes the minimum of 1 and $(n+1)D$. Using the methods of D.F. Votaw Jr.(1946),we show, in the third chapter, that the probability function

$f_n(x)$ of the random variable $\mu(X)$ is given by

$$(1.3)\quad f_n(x) = n! \sum_{j=1}^{q} \sum_{r=0}^{q-j} \frac{(-1)^r}{(n-j)!(j-1)!} \binom{n+1}{j}\binom{n-j+1}{r}(1-x)^{j-1}[x-D(j+r)]^{n-j},$$

where $qD \le x < (q+1)D, (q=1,2,\ldots,M; M = \text{minimum } (n,[1/D]))$, $x < m$, and where the symbol $[x]$ represents the greatest integer less than x. When $m = (n+1)D$, the distribution function, $F_n(x)$, of $\mu(X)$ has a discontinuous jump or saltus point corresponding to the finite probability that the unit interval will consist of exactly $n+1$ non-overlapping intervals $\{I\}$. At any such discontinuity of $F_n(x)$ the function $f_n(x)$, the probability function of $\mu(X)$, is not defined in the pure mathematical sense.

The expected value of $\mu(X)$ is found to be

$$(1.4)\qquad E(\mu(X)) = 1 - (1-D)^{n+1},$$

while its variance is given by

$$(1.5)\quad \mathrm{Var}\,(\mu(X)) = \begin{cases} \frac{1}{n+2}\left(2(1-D)^{n+2} + n(1-2D)^{n+2}\right) - (1-D)^{2n+2} & \text{for } 0 < D \le \frac{1}{2}, \\ \frac{2}{n+2}(1-D)^{n+2} - (1-D)^{2n+2} & \text{for } \frac{1}{2} < D < 1. \end{cases}$$

To our knowledge, the majority of the research which one might classify under the general heading of the measure of

a random set was carried out during the decade following a 1939 paper by W.L. Stevens. In the second chapter of this thesis, we present a review of some of the work of various authors who have conducted studies in this area. The remainder of this chapter is devoted to presenting, in chronological order beginning with Stevens' paper, the results obtained by these authors.

W.L. Stevens (1939) solved the following problem: consider $n+1$ arcs of length D $(0 < D < 1)$ at random points on a circle of unit circumference. Choose the end of one arc arbitrarily as the origin, and measure distances anti-clockwise around the circle, the location of each arc being represented by its first point, i.e. the point at the clockwise end. What is the probability that every point of the circle is included in at least one of the arcs, i.e. what is the probability that the circle is completely covered ? It is clear that this representation is equivalent to the representation of the linear problem described previously. Stevens proved that the required probability is given by

$$(1.6) \quad 1 - \binom{n+1}{1}(1-D)^n + \binom{n+1}{2}(1-2D)^n - \dots \pm \binom{n+1}{k}(1-kD)^n ,$$

where k is the greatest integer less than $1/D$. Stevens also calculated the frequency distribution of the number of

gaps (there is said to be a gap after the r th arc, if for a distance greater than D beyond the first point of this arc, there is no other arc) .

H.E. Robbins (1944) considered a similar but non-circular problem and proved the following

Theorem: Let X be a random Lebesgue measurable subset of Euclidean n-dimensional space E_n , with measure $\mu(X)$. For any point x of E_n let $p(x) = \Pr(x \in X)$. Suppose furthermore that the function

$$g(x,X) = \begin{cases} 1 & x \in X , \\ 0 & x \notin X \end{cases} \tag{1.7}$$

is a measurable function of the pair (x,X) . Then the expected value of the measure of X will be given by the Lebesgue integral of the function $p(x)$ over E_n , i.e.

$$E(\mu(X)) = \int_{E_n} p(x)\, d\mu(x) . \tag{1.8}$$

He also generalized this result by showing that the m th moment of $\mu(X)$ is given by

$$E(\mu^m(X)) = \int_{E_{mn}} p(x_1,x_2,\ldots,x_n)\, d\mu\,(x_1,x_2,\ldots,x_n) , \tag{1.9}$$

where $\mu(x_1,x_2,\ldots,x_n)$ denotes Lebesgue measure in E_{mn}, and where

$$(1.10)\quad p(x_1,x_2,\ldots,x_n) = P_r(x_1 \in X \text{ and } x_2 \in X \ldots \text{ and } x_n \in X) .$$

In the following problem, let the random set X be defined as follows: let $A_i, a_i (i = 1,2,\ldots,n)$ and δ be fixed positive numbers such that $a_i \leq 2\delta$; let R denote the n-dimensional interval consisting of all points $(x_1,x_2,\ldots,x_n)$ such that $0 \leq x_i \leq A_i$, and let R' denote the larger interval for which $-\delta \leq x_i \leq A_i + \delta$ (and also its measure $\prod_{i=1}^{n} (A_i + 2\delta))$. Let a fixed number N of intervals with sides parallel to the axes be chosen independently with the density function for the center of each interval constant and equal to $1/R'$ in R' . The set X is the intersection of the point set sum of the N intervals with R . What is the mean and variance of the measure of the set X ? In a second paper, H.E. Robbins (1945), using the main result of his 1944 paper, showed that

$$(1.11)\qquad E(\mu(X)) = R\left\{ 1 - \left(1 - \frac{r}{R'}\right)^{N} \right\} ,$$

where $r = \prod_{i=1}^{n} a_i$, and that

$$\text{(1.12)} \quad \operatorname{Var}(\mu(X)) = 2^n \int_{\langle a_n - A_n \rangle}^{a_n} \cdots \int_{\langle a_1 - A_1 \rangle}^{a_1} \left(1 - \frac{2r - \prod_{i=1}^{n} w_i}{R'} \right)^N \cdot \prod_{i=1}^{n} (w_i + A_i - a_i)\, dw_1 \cdots dw_n$$

$$+ \left(1 - \frac{2r}{R'}\right)^N \left\{ \prod_{i=1}^{n} A_i^2 - \prod_{i=1}^{n} \left(A_i^2 - \langle A_i - a_i \rangle^2\right) \right\} - R^2 \left(1 - \frac{r}{R'}\right)^{2N} ,$$

where the symbol $\langle x \rangle$ is defined by

$$\text{(1.13)} \qquad \langle x \rangle = \begin{cases} x & \text{if} \quad x \geq 0 , \\ 0 & \text{if} \quad x \leq 0 . \end{cases}$$

In this paper, Robbins also considered the more general case where the number N of n-dimensional intervals is not considered as being fixed but is taken as a random variable capable of assuming the values $0, 1, 2, \cdots$ with respective probabilities $p_o, p_1, p_2, \cdots$, and with generating function

$$\text{(1.14)} \qquad \varphi(t) = \sum_{N=0}^{\infty} p_N t^N \quad .$$

He showed that

$$E(\mu(X)) = R \left\{ 1 - \varphi\left(1 - \frac{r}{R'}\right) \right\} ,$$

and that

$$(1.16)\quad \mathrm{Var}\,(\mu(X)) = 2^n \int_{\langle a_n - A_n\rangle}^{a_n} \cdots \int_{\langle a_1 - A_1\rangle}^{a_1} \varphi\left(1 - \frac{2r - \prod_{i=1}^{n} w_i}{R}\right) \cdot \prod_{i=1}^{n} (w_i + A_i - a_i)\, dw_1 \cdots dw_n$$

$$+ \left\{ \prod_{i=1}^{n} A_i^2 - \prod_{i=1}^{n} (A_i^2 - \langle A_i - a_i \rangle^2) \right\} \varphi\left(1 - \frac{2r}{R'}\right) - R^2 \varphi^2\left(1 - \frac{r}{R'}\right)$$

Robbins also derived formulae for the corresponding problem for circles in the plane.

J. Bronowski and J. Neyman (1945), using methods quite different from those of Robbins, considered a two-dimensional problem and found the mean and variance of the measure of the random set X, where X was taken as the point set sum of rectangles in the plane (with sides parallel to the axes). Their results agreed with the results of Robbins (1945) for $n=2$ (see (1.11), (1.12), (1.15), and (1.16)).

In a later paper, L.A. Santaló (1947), using the methods of Robbins, found the mean and variance of the measure of the random set X consisting of the point set sum of N rectangles in the plane but having variable position. A similar problem was solved for n dimensions with the N rectangles replaced by spheres.

Returning to the one-dimensional case, D.F. Votaw Jr. (1946) considered a non-circular problem and derived the probability function of the random variable y defined as follows: consider a random sample $z_i (i=1,2,\dots,n)$ of n values of a one-dimensional random variable Z having distribution function $F(z)$. Arrange the n sample values in increasing order of magnitude as $x_i (i=1,2,\dots,n)$ and consider the set consisting of the intervals

$$(1.17) \qquad (x_i - \tfrac{1}{2}D \, , \, x_i + \tfrac{1}{2}D) \quad (0 < D < 1 \, ; \, i=1,2,\dots,n) \, .$$

Denote its measure by $\mu(X)$. Let $y = \mu(X) - D$. The range of y is $0 \leq y \leq m$, where m denotes the minimum of 1 and $(n-1)D$. Votaw showed that if $F(z)$ is given by (1.1), the probability function $f_n(y)$ of y is given by

$$(1.18) \quad f_n(y) = n \sum_{j=0}^{q} \sum_{r=0}^{q-j} (-1)^r \binom{n-1}{j} \binom{n-1}{j+1} \binom{n-j-1}{r} (1-y)^{j+1} [y-D(j+r)]^{n-j-2} \, ,$$

$$qD \leq y < (q+1)D \, , \, (q=0,1,\dots,M; M = \text{minimum } (n-2, \left[\tfrac{1}{D}\right])),$$

$$y < m \, .$$

The expected value of y is

$$E(y) = \frac{(n-1)}{(n+1)} \left[1 - (1-D)^{n+1} \right] \quad . \tag{1.19}$$

C. Domb (1947) considered random intervals with a constant length D ; whose left endpoints are random events distributed according to a Poisson distribution. He let $W(x,\beta)$ be the probability that the measure of the subset of the interval $(0,\beta)$ which is covered by random intervals does not exceed x . He then obtained the p-multiplied Laplace transform of $W(x,\beta)$ as the solution of an integral equation of the convolution type. From this he derived many results. In particular, he determined the probability that the interval $(0,\beta)$ is completely covered (or completely uncovered). He also derived an expansion of $W(x,\beta)$ and the moments of the $W(x,\beta)$ distribution. The method also yielded analogous results for the problem we have considered, namely that of random arcs on a circle.

CHAPTER II

REVIEW OF SOME RELATED PROBLEMS

As was mentioned earlier, this chapter is intended to furnish the reader with a review of the solutions of some of the problems that have evolved out of the general theme of the measure of a random set.

2.1. Stevens' Solution of the Problem of Coverage of a Circle of Unit Circumference.

W.L. Stevens (1939) considered $n+1$ points chosen independently and at random on the circle of unit circumference. To each point an arc of length D $(0 < D < 1)$ was attached, the point being at the clockwise end of the arc so that distances were measured anti-clockwise around the circle. He then posed the question: what is the probability that every point of the circle is included in at least one of the arcs, i.e. what is the probability that there does not occur a gap after any of the arcs (or, in terms of random sets, what is the probability that the random linear set formed by taking the intersection of the point set sum of the $n+1$ arcs with the unit circle is equal to one) ? He first proved

<u>Theorem 1</u>. Suppose that $n+1$ events (which are not independent) can each occur in two ways, denoted by H and T. The probability of exactly h H's and $n-h+1$ T's is

$$(2.1.1) \quad \binom{n+1}{h}\left\{f(h)-\binom{n+1-h}{1} f(h+1)+\binom{n+1-h}{2} f(h+2)+\dots+(-1)^{n+1-h} f(n+1)\right\},$$

where the probability that any h specified events are H, whatever be the remainder, is known to be $f(h)$ $(h=0,1,\dots,n+1)$. (Note that (2.1.1) may be written as

$$\binom{n+1}{h} p^h (1-p)^{n+1-h} ,$$

where, after expansion, we make the substitution $p^r = f(r)$).

<u>Proof</u>: Let $f(h,t)$ $(h,t=0,1,\dots,n+1;\ h+t \leq n+1)$ denote the probability that any h specified events are H, any other t specified events are T, and the remaining are either H or T. First, Stevens evaluated the probability $f(h,1)$ of h specified H's and one specified T. The admissible configurations are those with the h specified H's, excluding those which have an H at the event which is to be a T, i.e. excluding configurations with $h+1$ specified H's. Hence

$$f(h,1) = f(h) - f(h+1)$$
$$= -\Delta f(h) \quad .$$

Using a similar argument,

$$f(h,2) = f(h,1) - f(h+1,1)$$
$$= -\Delta f(h,1)$$
$$= \Delta^2 f(h) \quad .$$

Generalizing, he obtained

$$f(h,t) = (-\Delta)^t f(h)$$
$$= (1-E)^t f(h)$$

(2.1.2) $$= f(h) - \binom{t}{1} f(h+1) + \binom{t}{2} f(h+2) - \dots + (-1)^t f(h+t) \quad .$$

Assuming (2.1.2) to be true for all h and a particular t, it can be proved easily by induction that (2.1.2) is also true for all t. Hence, the probability that h specified events are H, and the remainder are all T, is

(2.1.3) $$f(h,n+1-h) = f(h) - \binom{n+1-h}{1} f(h+1) + \binom{n+1-h}{2} f(h+2) - \dots + (-1)^{n+1-h} f(n+1) \quad .$$

Now, the probability of exactly h unspecified H's and the remainder all T (and hence (2.1.1)) is obtained directly from equation (2.1.3) by noting that there are $\binom{n+1}{h}$ ways of selecting h events from $n+1$ distinguishable events, which completes the proof.

Next, Stevens considered the existence or non-existence of a gap after any specified arc to be the alternatives H and T respectively. On making this identification, it follows that the probability that a gap does not occur after any arc is given by (putting $h = 0$ into (2.1.1))

$$(2.1.4)\quad f(0) - \binom{n+1}{1} f(1) + \binom{n+1}{2} f(2) - \ldots + (-1)^{n+1} f(n+1) \quad ,$$

where $f(i)$ is the probability that there are gaps in i specified places (and in the remaining places there may or may not be a gap). Therefore, in order to obtain the required probability, we need only determine $f(i)$. Stevens began by finding the probability $f(1)$ that there is a gap after the r th arc. The restriction that there shall occur a gap after the r th arc is equivalent to the restriction that no arc shall occur in the portion of the circumference of length D terminating at 1. For any configuration obeying the former restriction may be converted to one obeying the latter restriction by moving arcs $r+1, r+2, \ldots, n+1$ clockwise through

a distance D. Conversely, the latter may be converted to the former by moving arcs $r+1, r+2, \ldots, n+1$ anti-clockwise through a distance D. The required probability is therefore the probability that the n arcs, excluding the first, shall fall in a portion of the circumference of length $1-D$, i.e.

$$f(1) = (1-D)^n .$$

Notice that, in keeping with the definition of $f(h)$ used in the theorem, the possible configurations contributing to this probability include those which contain gaps elsewhere than in the specified place. In the same way, he found the simultaneous probability $f(2)$ of gaps after r_1 and r_2, where $r_2 > r_1$, as

$$f(2) = (1-2D)^n .$$

(Since the restriction that there be gaps after r_1 and r_2 is equivalent to the restriction that no arc shall occur in the portion of the circumference of length $2D$ terminating at 1; for a configuration of the former kind can be converted to one of the latter by moving arcs $r_1+1, r_1+2, \ldots, r_2$ clockwise through a distance of D, and arcs $r_2+1, r_2+2, \ldots, n+1$ clockwise through a distance of $2D$, or conversely. As before, configurations having gaps in the specified places

and elsewhere are included in the above probability). The generalization is now obvious. If there are gaps in i specified places, the distances between each pair of arcs enclosing a gap may be reduced by D, to produce a configuration with the portion of length iD terminating at 1 containing no arcs; and conversely. Hence

$$(2.1.5) \qquad f(i) = \begin{cases} (1-iD)^n & i \le k , \\ 0 & i > k , \end{cases}$$

where k is the greatest integer less than $1/D$.

Therefore, from equations (2.1.4) and (2.1.5), the probability that every point on the circumference of the circle is included in at least one of the arcs is

$$(2.1.6) \quad 1 - \binom{n+1}{1}(1-D)^n + \binom{n+1}{2}(1-2D)^n - \ldots + (-1)^k\binom{n+1}{k}(1-kD)^n .$$

Notice that from equations (2.1.1) and (2.1.5), we may write down immediately the frequency distribution of the number of gaps. For example, the probability that there are exactly j gaps $(j = 0,1,\ldots,k)$ is

$$(2.1.7) \quad \binom{n+1}{j}\Big\{(1-jD)^n - \binom{n+1-j}{1}(1-(j+1)D)^n + \binom{n+1-j}{2}(1-(j+2)D)^n + \ldots + (-1)^{k-j}\binom{n+1-j}{k-j}(1-kD)^n\Big\} .$$

2.2. Moments of the Measure of a Random Set.

In the first of two papers, H.E. Robbins (1944) derived formulae for the moments of the measure of any random set, and applied these formulae to find the mean and variance of a random point set sum of intervals on the real line. These formulae were used by Robbins (1945) and others to obtain more generalized results.

Robbins' most important contribution was the proof of

Theorem 2. Let X be a random Lebesgue measurable subset of Euclidean n-dimensional space E_n, with measure $\mu(X)$. For any point x of E_n let $p(x) = Pr(x \in X)$. Suppose, furthermore that the function

$$(2.2.1) \qquad g(x,X) = \begin{cases} 1 & \text{for} \quad x \in X, \\ 0 & \text{for} \quad x \notin X \end{cases}$$

is a measurable function of the pair (x,X). Then the expected value of the measure of X, $\mu(X)$, will be given by the Lebesgue integral of the function $p(x)$ over E_n, i.e.

$$(2.2.2) \qquad E(\mu(X)) = \int_{E_n} p(x)\, d\mu(x) .$$

Proof: Denote by T the space of all possible values of X and let there be defined a probability measure $\rho(X)$ on T so that, for every ρ-measurable subset S of T, the probability that X shall belong to S is given by the Lebesgue - Stieltjes integral

$$(2.2.3) \qquad \Pr(X \in S) = \int_T C_s(X)\, d\rho(X) ,$$

where the integrand is the characteristic function of S,

$$(2.2.4) \qquad C_s(X) = \begin{cases} 1 & \text{for} \quad X \in S , \\ 0 & \text{for} \quad X \notin S . \end{cases}$$

For every point x of E_n, Robbins defined $S(x)$ as the set of all X in T which contain the point x. Then for every point x in E_n we have from (2.2.1) and (2.2.4)

$$(2.2.5) \qquad g(x,X) = C_{S(x)}(X) .$$

Denoting the Lebesgue measure in E_n of the set X by $\mu(X)$, we have

$$E(\mu(X)) = \int_T \mu(X)\, d\rho(X) \,, \tag{2.2.6}$$

and, from (2.2.1), we have

$$\mu(X) = \int_X d\mu(x) = \int_{E_n} g(x,X)\, d\mu(x) \,. \tag{2.2.7}$$

Now, assuming that the function $g(x,X)$ is a $\mu \times \rho$ - measurable function of the pair (x,X) in the product space of E_n with T, it follows from Fubini's theorem and from (2.2.6) and (2.2.7) that

$$E(\mu(X)) = \int_T \int_{E_n} g(x,X)\, d\mu(x) d\rho(X)$$

$$= \int_{E_n} \int_T g(x,X)\, d\rho(X) d\mu(x) = \int_{E_n \times T} g(x,X)\, d\mu\rho(x,X) . \tag{2.2.8}$$

But, from (2.2.1) and (2.2.3),

$$\int_T g(x,X)\, d\rho(X) = \Pr(X \in S(x)) = \Pr(x \in X) \,, \tag{2.2.9}$$

and hence (2.2.2) follows, i.e.

$$E(\mu(X)) = \int_{E_n} \Pr(x \in X)\,d\mu(x) \ .$$

Robbins next generalized the above fundamental result to obtain similar expressions for the higher moments of $\mu(X)$. The formula for the m th moment of $\mu(X)$ was given as

$$(2.2.10) \qquad E(\mu^m(X)) = \int_{E_{mn}} p(x_1,x_2,\ldots,x_m)\,d\mu(x_1,x_2\ldots,x_m) \ ,$$

where $\mu(x_1,x_2,\ldots,x_m)$ denotes Lebesgue measure in E_{mn} and where

$$p(x_1,x_2,\ldots,x_m) = \Pr(x_1 \in X \text{ and } x_2 \in X \ldots \text{ and } x_m \in X) \ .$$

We are now in a position to apply the above results of Robbins to the random set X described at the beginning of Chapter 1 . Recall that we can define this random set X as follows: $n+1$ points are chosen independently and at random on the circle of unit circumference. To each point an arc of length $D\,(0 < D < 1)$ is attached, the point being at the clockwise end of the arc. The set X is that portion of the circle of unit circumference that is covered by the

$n+1$ arcs. Now, if a set is measurable, then its characteristic function is a measurable function. Hence, since the set X defined above is a measurable set, the conditions of Theorem 2 are clearly satisfied. The probability that any point x on the circle of unit circumference shall be contained in the i th arc is clearly D. Hence

$$
\begin{aligned}
(2.2.11) \quad \Pr(x \in X) &= \Pr(x \text{ belongs to at least one of the } n+1 \text{ arcs}) \\
&= 1 - \Pr(x \text{ does not belong to any of the } n+1 \text{ arcs}) \\
&= 1 - (1-D)^{n+1} \quad ,
\end{aligned}
$$

so that, for any point x on the circle of unit circumference,

$$(2.2.12) \qquad \Pr(x \in X) = 1 - (1-D)^{n+1} \quad .$$

From (2.2.2) it follows that

$$(2.2.13) \qquad E(\mu(X)) = \int_0^1 (1-(1-D)^{n+1})\, dx = 1-(1-D)^{n+1} \quad .$$

To evaluate $E(\mu^2(X))$ we make use of the identity

$$(2.2.14) \quad \Pr(A \text{ and } B) = \Pr(A) + \Pr(B) + \Pr(\text{neither } A \text{ nor } B) - 1 \quad ,$$

which holds for any two events A and B. It follows from (2.2.12) and (2.2.14) that if x and y are any two points on the circle of unit circumference, then

$$(2.2.15)\quad \begin{aligned} p(x,y) &= \Pr(x \in X \text{ and } y \in X) \\ &= \Pr(x \in X) + \Pr(y \in X) + \Pr(x \notin X \text{ and } y \notin X) - 1 \\ &= 1 - 2(1-D)^{n+1} + \Pr(x \notin X \text{ and } y \notin X) . \end{aligned}$$

Let

$$(2.2.16)\qquad h(x,y) = \Pr(x \notin X \text{ and } y \notin X) ,$$

and assume, without loss of generality, that $y > x$. From (2.2.10), (2.2.15), and (2.2.16), we want to evaluate

$$(2.2.17)\quad \begin{aligned} E(\mu^2(X)) &= \int_0^1 \int_0^1 p(x,y)\, dy\, dx \\ &= \int_0^1 \int_0^1 [1-2(1-D)^{n+1} + h(x,y)]\, dy\, dx \\ &= 1-2(1-D)^{n+1} + 2 \int_0^1 \int_x^1 h(x,y)\, dy\, dx . \end{aligned}$$

For $D \leq \frac{1}{2}$, it may be shown that

$$(2.2.18)\quad h(x,y) = \begin{cases} (1-(D+y-x))^{n+1} & \text{for } \quad y-x \le D \;, \\ (1-2D)^{n+1} & \text{for } \; D < y-x < 1-D \;, \\ (1-(D+x+1-y))^{n+1} & \text{for } \quad y-x \ge 1-D \;. \end{cases}$$

Hence, from (2.2.17), when the latter integral in that expression is evaluated, we obtain

$$E(\mu^2(X)) = 1-2(1-D)^{n+1} + \frac{1}{n+2}\left(2(1-D)^{n+2} + n(1-2D)^{n+2}\right) \quad (D \le \tfrac{1}{2}) \;.$$

Combining this with (2.2.13), we obtain the variance of $\mu(X)$ as

$$(2.2.19)\quad \mathrm{Var}\,(\mu(X)) = E(\mu^2(X)) - E^2(\mu(X))$$
$$= \frac{1}{n+2}\left(2(1-D)^{n+2} + n(1-2D)^{n+2}\right) - (1-D)^{2n+2} \quad (D \le \tfrac{1}{2}) \;.$$

If $D > \frac{1}{2}$, it can be shown that

$$(2.2.20)\quad h(x,y) = \begin{cases} (1-(D+y-x))^{n+1} & \text{for } \quad y-x \le 1-D \;, \\ 0 & \text{for } \; 1-D < y-x < D \;, \\ (1-(D+x+1-y))^{n+1} & \text{for } \quad y-x \ge D \;, \end{cases}$$

from which, together with (2.2.17), we obtain

$$E(\mu^2(X)) = 1-2(1-D)^{n+1} + 2(1-D)^{n+2} \qquad (D > \tfrac{1}{2}),$$

and

$$(2.2.21) \quad \mathrm{Var}\,(\mu(X)) = \frac{2}{n+2}(1-D)^{n+2} - (1-D)^{2n+2} \qquad (D > \tfrac{1}{2}).$$

2.3 "The Probability Distribution of the Measure of a Random Linear Set"

Consider now the problem of determining the probability function of the measure of a one-dimensional random set. Let n points $z_i\,(i=1,2,\ldots,n)$ be selected independently and at random from the interval $(0,1)$, the distribution of any z_i being the uniform distribution with distribution function given by (1.1), i.e.

$$F(z) = z \qquad (0 \leq z \leq 1) \quad .$$

Let these n points be arranged in increasing order of magnitude as $x_i\,(i=1,2,\ldots,n)$, and with each x_i let there be associated an interval of length $D(0 < D < 1)$ centered at x_i. Let $\mu(X)$ denote the measure of the set X consisting of the point set sum of the intervals

$$(x_i - \tfrac{1}{2}D\,,\ x_i + \tfrac{1}{2}D) \qquad (0 < D < 1\,;\ i=1,2,\ldots,n)\ .$$

We now present in detail a paper by D.F. Votaw Jr. (1946), in which he calculated the probability function of the random variable $\mu(X)$. This is a very useful and important paper in that the general method used by Votaw in this paper leads us to the solution of the similar one-dimensional but circular problem that was described in the introductory chapter.

Votaw began by making the transformation

$$y_o = x_1$$

$$y_i = x_{i+1} - x_i \qquad (i = 1, 2, \ldots, n-1) ,$$

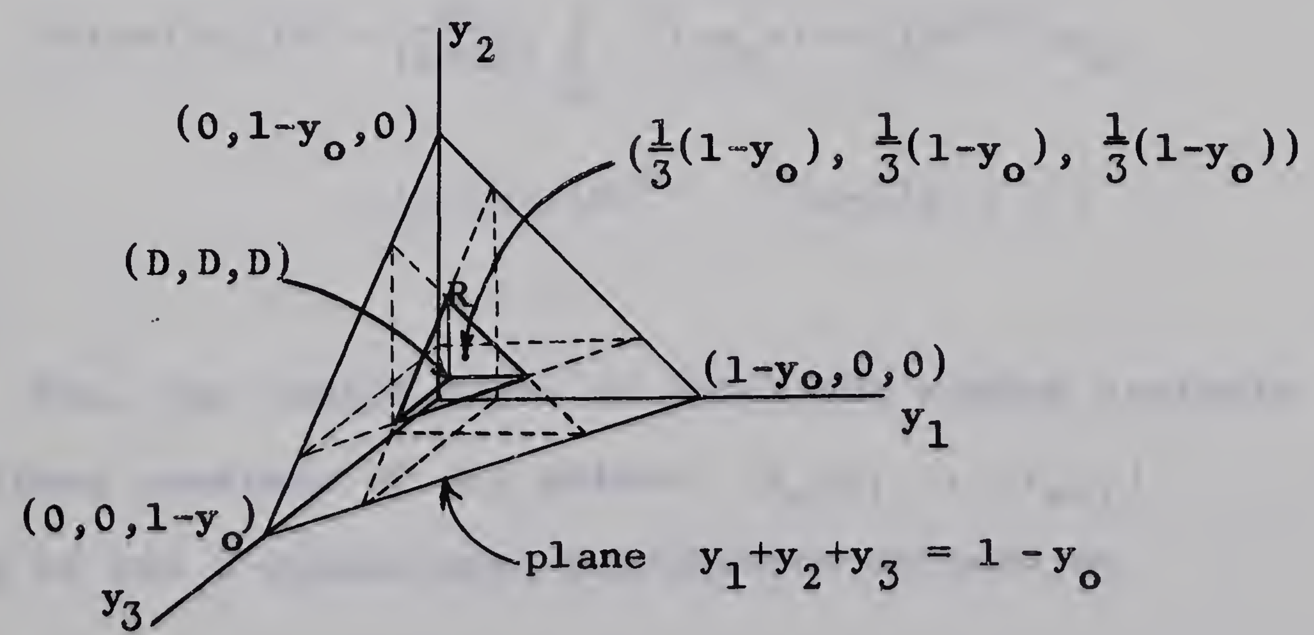
and letting $y = \mu(X) - D$. The range of y becomes $0 \leq y \leq m$, where m denotes the minimum of 1 and $(n-1)D$. It is clear that y may be expressed as

$$(2.3.1) \qquad y = \sum_{i=1}^{n-1} m(y_i, D) ,$$

where $m(y_i, D)$ denotes the minimum of y_i and D. From the theory of order statistics it can be shown that the joint density function of $x_1, x_2, \ldots, x_n$ is $n!\,(0 < x_1 < x_2 < \ldots < x_n < 1)$, and since $\partial(y_o, y_1, \ldots, y_{n-1})/\partial(x_1, x_2, \ldots x_n) = 1$, it is clear that the joint density function of $y_o, y_1, \ldots, y_{n-1}$ is $n!$ $(y_i \geq 0,\ i = 0, 1, \ldots n-1;\ \sum_{i=0}^{n-1} y_i \leq 1)$.

If $m = (n-1)D$, it is clear that there will be a finite probability that the set X will consist of the union of n disjoint intervals of length D and hence, the probability function of y will have a saltus point at $y = (n-1)D$. Now, given that $m = (n-1)D$, $y = (n-1)D$ if and only if $y_i \geq D$ $(i = 1,2,\ldots,n-1)$. In order to determine the amount of the saltus at $y = (n-1)D$, Votaw first fixed y_o and calculated the volume of the $(n-1)$-dimensional region in which any point $(y_o,\ldots,y_{n-1})$ satisfies the condition $y_1 \geq D, y_2 \geq D, \ldots, y_{n-1} \geq D$. Recalling that we must have $\sum_{i=0}^{n-1} y_i \leq 1$, this region is clearly R in the figure below, and its volume, V say, is given by

$$V = \int_0^{1-y_o} \int_0^{1-y_o-y_1} \cdots \int_0^{1-y_o-y_1-\cdots-y_{n-2}} dy_{n-1}\cdots dy_2 dy_1 .$$



Now, making the substitution $z_i = \dfrac{y_i - D}{1-y_o-(n-1)D}$ $(i = 1,2,\ldots,n-1)$,

so that $\sum_{i=1}^{n-1} z_i \le 1$ $(z_i \ge 0,\ i=1,2,\ldots,n-1)$, we have

$$V = (1-y_o-(n-1)D)^{n-1} \int_0^1 \int_0^{1-z_1} \cdots \int_0^{1-z_1-z_2-\cdots-z_{n-2}} dz_{n-1} \cdots dz_2\, dz_1 ,$$

which, by Dirichlets integral, becomes

$$V = \frac{(1-y_o-(n-1)D)^{n-1}}{(n-1)!} .$$

By letting y_o vary, it is clear from the figure above that

$$(2.3.2) \qquad Pr(y=(n-1)D) = \frac{n!}{(n-1)!} \int_0^{1-(n-1)D} (1-y_o-(n-1)D)^{n-1} dy_o$$

$$= (1-(n-1)D)^n \qquad (n-1)D \le 1 .$$

Now, the sample space on which the random variable y is defined consists of all points $(y_o, y_1, \ldots, y_{n-1})$ belonging to the n-dimensional tetrahedron defined by

$y_i \ge 0$ $(i = 0,1,\ldots,n-1)$ and $\sum_{i=0}^{n-1} y_i \le 1$. As we shall see

later, the probability that $Y < y < Y + \Delta Y$ (where $Y < m$ and ΔY denotes an arbitrary small positive increment in Y) can be evaluated by determining volumes of certain regions contained in this tetrahedron. To provide a starting point for the derivation of the probability function of y, Votaw considered the following conditions.

(a) $qD \leq Y < (q+1)D$ ($q = 0,1,\ldots,M$; M denotes the minimum of $(n-2)$ and the greatest integer less than $1/D$).

For any particular value of q, he decomposed the set $y_o, y_1, \ldots, y_{n-1}$ into three subsets (y_o), $(y_1, y_2, \ldots, y_j)$, and $(y_{j+1}, y_{j+2}, \ldots, y_{n-1})$ such that with y_o fixed

(b) $y_u \geq D$ ($u = 1,2,\ldots,j$; $j \leq q$),

(c) $y_v < D$ ($v = j+1, j+2, \ldots, n-1$).

Now, since $\sum_{i=0}^{n-1} y_i \leq 1$, we have

$$\sum_{u=0}^{j} y_u + \sum_{v=j+1}^{n-1} y_v \leq 1 ,$$

and, as a consequence of (b) and (c) we have, from equation (2.3.1)

$$y = jD + \sum_{v=j+1}^{n-1} y_v \quad ;$$

thus

$$\sum_{u=0}^{j} y_u + y - jD \leq 1 \quad .$$

From this, we have

(d) $$\sum_{u=1}^{j} y_u \leq 1 - y_o - y + jD \quad .$$

Now, let $A_j(y,y_o)$ denote the volume of the j-dimensional region in which any point $(y_1,y_2,\ldots,y_j)$ satisfies conditions (b) and (d) , and let $B_{n-j-2}(y)$ denote the volume of the (n-j-2)-dimensional region which is the intersection of the hyperplane $\sum_{v=j+1}^{n-1} y_v = y-jD$ with the (n-j-1)-dimensional cube given by condition (c) .

Let us first evaluate $A_j(y,y_o)$. For convenience, let $h = 1-y_o-y+jD$. Then from conditions (b) and (d) , this volume is simply (see derivation of (2.3.2))

$$A_j(y,y_o) = \int_D^h \int_D^{h-y_1} \cdots \int_D^{h-y_1-y_2-\cdots-y_{j-1}} dy_j \cdots dy_2 dy_1 \quad .$$

Letting $y_1 = h\xi_1, y_2 = h\xi_2, \ldots, y_j = h\xi_j$, we have

$$A_j(y,y_o) = h^j \int_{D/h}^{1} \int_{D/h}^{1-\xi_1} \cdots \int_{D/h}^{1-\xi_1-\xi_2-\cdots-\xi_{j-1}} d\xi_j \cdots d\xi_2 d\xi_1 \quad .$$

Now, on making the substitution $z_i = \dfrac{\xi_i - \frac{D}{h}}{1 - j\frac{D}{h}}$ we have

$\sum_{i=1}^{j} z_i \leq 1$ and $z_i \geq 0$ $(i = 1,2,\ldots,j)$, so that

$$A_j(y,y_o) = h^j (1 - \frac{jD}{h})^j \int_0^1 \int_0^{1-z_1} \cdots \int_0^{1-z_1-z_2-\cdots-z_{j-1}} dz_j \cdots dz_2 dz_1$$

(2.3.3) $$= \frac{(h-jD)^j}{j!} = \frac{(1-y_o-y)^j}{j!} \quad .$$

Note that the range of y_o is $0 < y_o < 1-y$. To evaluate $B_{n-j-2}(y)$, Votaw used a result from a paper by P. Hall (1927), in which it is proved that the volume cut from the hyperplane

$\sum_{i=1}^{N} x_i = p$ by the unit cube $0 \leq x_i \leq 1$ $(i = 1, 2, \ldots, N)$ is given by

$$(2.3.4) \qquad v(p) = \frac{\sqrt{N}}{(N-1)!} \sum_{r=0}^{\kappa} (-1)^r \binom{N}{r} (p-r)^{N-1} ,$$

where κ is equal to the greatest integer less than p.

Now, we wish to determine the volume cut from the hyperplane

$$\sum_{v=j+1}^{n-1} y_v = y - jD$$

by the hypercube $0 \leq y_v \leq D$ $(v = j+1, \ldots, n-1)$. Let $y_v = z_{v-j}$ $(v=j+1, \ldots, n-1)$, then we want the volume cut from the hyperplane

$$\sum_{i=1}^{n-j-1} z_i = y - jD$$

by the hypercube $0 \leq z_i \leq D$ $(i=1,2,\ldots,n-j-1)$. In order to apply Hall's result, let us transform the above hypercube to the unit hypercube by letting $x_i = z_i/D$. Since $0 < D < 1$, this transformation is an expansion, so we can find the volume cut from the $(n-j-2)$-dimensional hyperplane

$$\sum_{i=1}^{n-j-1} x_i = \frac{y}{D} - j$$

by the unit hypercube $0 \le x_i \le 1$ $(i=1,2,\ldots,n-j-1)$ using Hall's result and then multiply the result by D^{n-j-2}.

Hence, using Hall, we have $p = \frac{y}{D} - j$, $N = n-j-1$, and since

$$qD \le y < (q+1)D$$

$$q-j \le \frac{y}{D} - j < (q+1) - j$$

$$q-j \le p < (q+1) - j ,$$

we have $\kappa = q-j$, so that, from (2.3.4)

$$B_{n-j-2}(y) = D^{n-j-2} \frac{\sqrt{n-j-1}}{(n-j-2)!} \sum_{r=0}^{q-j} (-1)^r \binom{n-j-1}{r} \left(\frac{y}{D} - j - r\right)^{n-j-2}$$

(2.3.5)
$$= \frac{\sqrt{n-j-1}}{(n-j-2)!} \sum_{r=0}^{q-j} (-1)^r \binom{n-j-1}{r} \left[y - D(j+r)\right]^{n-j-2} .$$

$A_j(y,y_o)$ and $B_{n-j-2}(y)$ define contents of regions in orthogonal subspaces of the $(n-1)$-dimensional tetrahedron $\sum_{i=1}^{n-1} y_i \le 1 - y_o$. The content of the $j + (n-j-2) = n-2$

dimensional region satisfying conditions (b) , (c) , and (d) is therefore $A_j(y,y_o)\, B_{n-j-2}(y)$. Therefore, the probability that $Y < y < Y + \Delta Y$ and that conditions (b) , (c) , and (d) are satisfied is

$$(2.3.6) \qquad n! \int_{y=Y}^{Y+\Delta Y} B_{n-j-2}(y) \int_{y_o=0}^{1-y} A_j(y,y_o)\, dy_o \; \frac{dy}{\sqrt{n-j-1}} \,.$$

To illustrate why the factor $1/\sqrt{n-j-1}$ occurs in (2.3.6) , consider the following example. Suppose we let $n = 7$, $D = \frac{1}{6}$, and suppose we take a particular case where $q=4$ and $j = 3$. Hence, conditions (a) through (d) become

(a) $\frac{4}{6} \leq Y < \frac{5}{6}$,

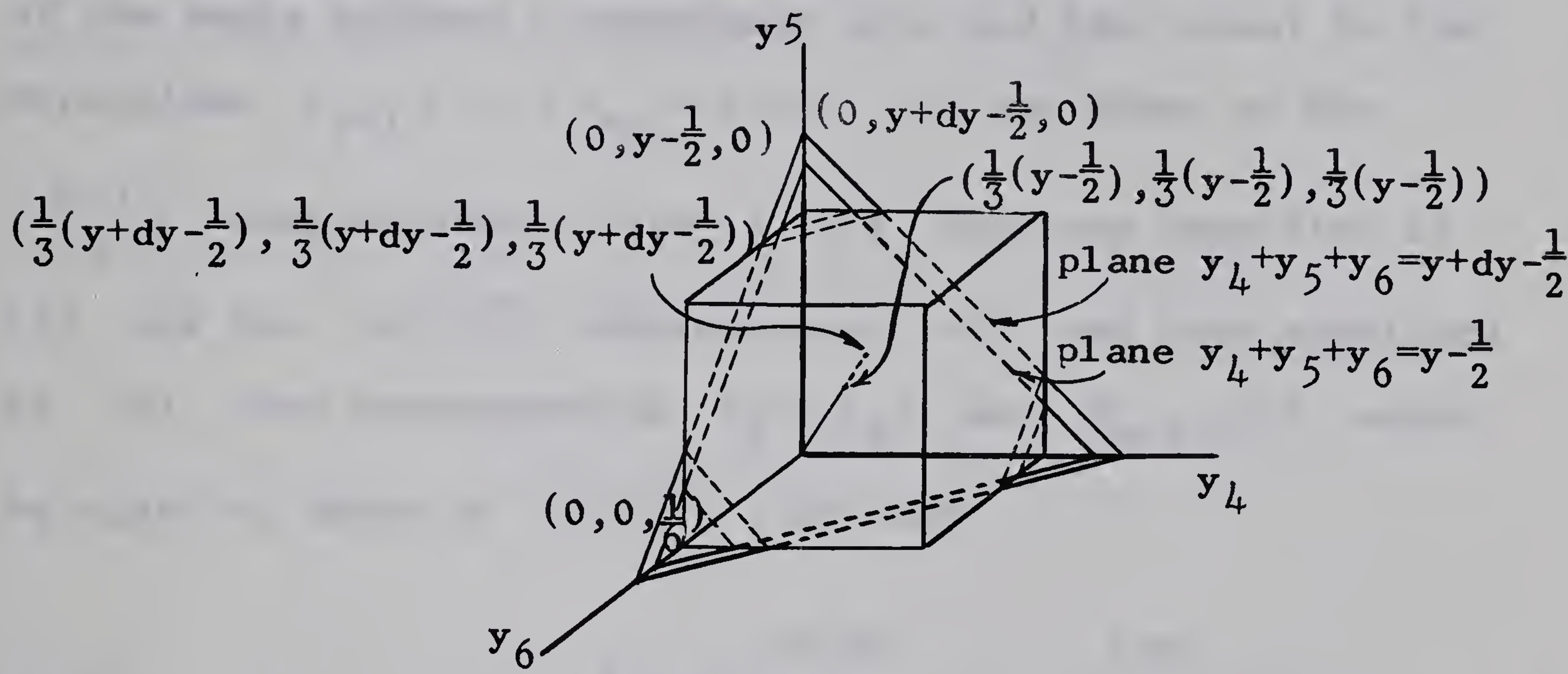
(b) $y_1 \geq \frac{1}{6}, y_2 \geq \frac{1}{6}, y_3 \geq \frac{1}{6}$,

(c) $y_4 < \frac{1}{6}, y_5 < \frac{1}{6}, y_6 < \frac{1}{6}$,

(d) $y_1 + y_2 + y_3 \leq 1 - y_o - y + \frac{1}{2}$,

and we also have $y_4 + y_5 + y_6 = y - \frac{1}{2}$. Now, $B_2(y)$ is equal to the content of the 2-dimensional region containing all points (y_4,y_5,y_6) which satisfy condition (c) and

the equation $\sum_{v=4}^{6} y_v = y - \frac{1}{2}$, i.e. $B_2(y)$ is the content of the plane $\sum_{v=4}^{6} y_v = y - \frac{1}{2}$ cut off by a cube (at the origin) of side $\frac{1}{6}$.



We want to evaluate the probability that $Y \leq y < Y + \Delta Y$ and that conditions (a) through (d) are satisfied. This is given by equation (2.3.6) which involves the volume between the planes $\sum_{v=4}^{6} y_v = y - \frac{1}{2}$ and $\sum_{v=4}^{6} y_v = y + dy - \frac{1}{2}$. To determine this volume, we need only multiply $B_2(y)$ by the perpendicular distance between the planes (this is assuming, of course, that dy is small enough that we can neglect the volume gained by

not considering the effect of the boundary of this region with the cube). This distance, dz say, is simply

$dz = \left\{3\left(\frac{dy}{3}\right)^2\right\}^{\frac{1}{2}} = \frac{dy}{\sqrt{3}}$. Hence, the volume of the region is

$B_2(y)\,dz = B_2(y)\,\frac{dy}{\sqrt{3}}$. In general $1/\sqrt{n-j-1}$ is the cosine of the angle between a coordinate axis and the normal to the hyperplane $y_{j+1} + \ldots + y_{n-1} = y - jD$. If any other of the $\binom{n-1}{j}$ combinations of the j y's had been specified in (b) and the $(n-j-1)$ complementary y's had been specified in (c) , the corresponding $A_j(y, y_o)$ and $B_{n-j-2}(y)$ would be equal to those in (2.3.6) , so that

$$\Pr(Y < y < Y + \Delta Y) = n! \sum_{j=0}^{q} \binom{n-1}{j} \int_{y=Y}^{Y+\Delta Y} B_{n-j-2}(y) \int_{y_o=0}^{1-y} A_j(y, y_o)\,dy_o \, \frac{dy}{\sqrt{n-j-1}}$$

for $\qquad qD \leq Y < (q+1)D$, $Y < m$, $(q = 0, 1, \ldots, M)$.

Therefore, from (2.3.3) and (2.3.5) we have

$$\Pr(Y < y < Y + \Delta Y) = n! \sum_{j=0}^{q} \binom{n-1}{j} \int_{Y}^{Y+\Delta Y} \frac{1}{(n-j-2)!} \sum_{r=0}^{q-j} (-1)^r \binom{n-j-1}{r} \left[y - D(j+r)\right]^{n-j-2}$$

$$\cdot \int_{0}^{1-y} \frac{(1-y_o-y)^j}{j!}\, dy_o\, dy$$

$$= n \sum_{j=0}^{q} \sum_{r=0}^{q-j} (-1)^r \binom{n-1}{j} \binom{n-j-1}{r} \binom{n-1}{j+1}$$

$$\cdot \int_{Y}^{Y+\Delta Y} \left[y-D(j+r)\right]^{n-j-2} (1-y)^{j+1} \, dy \ .$$

Applying the mean value theorem, there exists a ξ such that

$$\frac{\Pr(Y < y < Y + \Delta Y)}{\Delta Y} = n \sum_{j=1}^{q} \sum_{r=0}^{q-j} (-1)^r \binom{n-1}{j} \binom{n-j-1}{r} \binom{n-1}{j+1}$$

$$\cdot \left[\xi - D(j+r)\right]^{n-j-2} (1-\xi)^{j+1} \quad ,$$

where $Y < \xi < Y + \Delta Y$. Letting $\Delta Y \to 0$, we have the probability function of y as

$$(2.3.7) \quad f_n(y) = n \sum_{j=1}^{q} \sum_{r=0}^{q-j} (-1)^r \binom{n-1}{j} \binom{n-j-1}{r} \binom{n-1}{j+1}$$

$$\cdot \left[y-D(j+r)\right]^{n-j-2} (1-y)^{j+1} \quad ,$$

$$qD \le y < (q+1)D \quad , \quad (q = 0,1,\ldots,M) \quad , \quad y < m \ .$$

Votaw next noted that if $(n-1)D < 1$, then $f_n(y)$ is not defined at $y = (n-1)D$ (hence the range of definition of $f_n(y)$ is given as $0 \le y < m$). This is due to the fact

that there is a saltus of amount $(1-(n-1)D)^n$ at $y=(n-1)D$ (see (2.3.2)). If $m=1$, the range of y in (2.3.7) is $0 \le y \le 1$. He then stated that the probability function $f_n(y)$ is continuous over the range $0 \le y < m$ with the exception of a simple discontinuity at $y=(n-2)D$ in the case $n \ge 3$ and $(n-2)D < 1$. The distribution function of y is continuous with the exception, in the case $(n-1)D < 1$, of a discontinuity corresponding to the saltus of amount $(1-(n-1)D)^n$ at $y=(n-1)D$.

The expected value of y (and hence of $\mu(X)$) can be derived in the usual way from $f_n(y)$ or by the use of a result of Robbins. Here, we present the latter method. Recall that the expected value of the measure of a random Lebesgue measurable subset X of Euclidean n-dimensional space E_n, is given by the Lebesgue integral

$$E(\mu(X)) = \int_{E_n} p(x)\,dx ,$$

where, for any point x of E_n, $p(x) = Pr(x \in X)$, provided the function

$$g(x,X) = \begin{cases} 1 & \text{for } x \in X , \\ 0 & \text{for } x \notin X , \end{cases}$$

is a measurable function of the pair (x,X). Let x be any point belonging to the interval $(-D/2\ ,\ 1+D/2)$, and let p, say, be the probability that x shall be contained in the i th subinterval of length D centered at $x_i (i=1,2,\ldots,n-1)$. It is clear, then, that

$$p = \begin{cases} x + \frac{D}{2} & x \in (-\frac{D}{2}, 0) \\ x + \frac{D}{2} & x \in (0, \frac{D}{2}) \\ D & x \in (\frac{D}{2}, 1-\frac{D}{2}) \\ 1-x+\frac{D}{2} & x \in (1-\frac{D}{2}, 1) \\ 1-x+\frac{D}{2} & x \in (1, 1+\frac{D}{2}), \end{cases}$$

or

$$p = \begin{cases} x + \frac{D}{2} & -\frac{D}{2} \leq x < \frac{D}{2} \\ D & \frac{D}{2} \leq x < 1-\frac{D}{2} \\ 1-x+\frac{D}{2} & 1-\frac{D}{2} \leq x \leq 1+\frac{D}{2} . \end{cases}$$

Now, from (2.2.11), $p(x) = \Pr(x \in X) = 1-(1-p)^n$, hence, for $0 < D < 1$

$$E(\mu(X)) = \int_{-D/2}^{D/2} \left[1-(1-x-D/2)^n\right] dx + \int_{D/2}^{1-D/2} \left[1-(1-D)^n\right] dx + \int_{1-D/2}^{1+D/2} \left[1-(x-D/2)^n\right] dx$$

(2.3.8) $$= D + \frac{n-1}{n+1}\,(1-(1-D)^{n+1})\ .$$

2.4 Random Intervals on a Line

C. Domb (1947) considered a sequence of random events occuring as points on the line t from $-\infty$ to $+\infty$. On assuming that the probability of the occurrence of two or more of these points in a very short interval $(t\,,\,t + dt)$ is $\lambda\,dt$, one can show that the probability of finding exactly j points (events) within a fixed interval of length β say, is given by the Poisson distribution, with parameter λ, i.e.

$$p(j\,,\,\lambda\,\beta\,) = \frac{e^{-\lambda\beta}(\lambda\beta)^j}{j!}\ .$$

Domb considered the case where each event (or occurrence of a point) consists of an interval of length D (a positive constant); an event being characterized by its first point. He let $W(x\,,\,\beta)$ be the differential coefficient (with respect to x) of the function $T(x\,,\,\beta)$ defined to be equal to the probability that the covered portion of the interval $(0\,,\,\beta)$ is less than or equal to x. He then obtained the function

(2.4.1) $$G(p,z) = e^{\lambda z}\,\mathcal{L}\,\{W(x\,,\beta)\}\ ,$$

where $z = \beta - x$ and $\mathcal{L}\{f(x,z)\} = G(p,z)$ denotes the p-multiplied Laplace transform of $f(x,z)$, i.e

$$(2.4.2) \qquad p\int_0^{\infty} e^{-px} f(x,z)\,dx = G(p,z),$$

as the solution of an integral equation of the convolution type. From this solution he was able to derive the following results. The probability that the total interval $(0,\beta)$ is covered is given by the function

$$(2.4.3) \qquad z(\beta) = 1 - e^{-\lambda D}(1+\lambda\beta) + e^{-2\lambda D}\left[\lambda(\beta-D) + \frac{\lambda^2(\beta-D)^2}{2!}\right] - \dots$$

$$+ (-1)^{r+1} e^{-(r+1)\lambda D}\left[\lambda^r \frac{(\beta-rD)^r}{r!} + \lambda^{r+1}\frac{(\beta-rD)^{r+1}}{(r+1)!}\right] + \dots,$$

all terms beyond $\beta - nD$ being ignored if $nD < \beta \le (n+1)D$.

The probability of the interval being completely uncovered is $e^{-\lambda(\beta-D)}$ while the probability that r non-overlapping events occur in the interval (including no overlapping at the end-points) is

$$(2.4.4) \qquad \lambda^r(\beta-rD)^r \frac{e^{-\lambda(\beta+D)}}{r!} \qquad (\beta \ge rD).$$

Next, he derived an expansion of $W(x,\beta)$ according to the number of events in the interval $(0, \beta)$; the complete solution of which, he admitted as very difficult. However, he succeeded in deriving an expansion which gives the probability that given $n+1$ events occur in $(-D, \beta)$, the interval $(0, \beta)$ is completely covered. Using this expression he was able to obtain a result of Stevens, namely (2.1.6) .

CHAPTER III

MAIN RESULTS

Consider $n+1$ points $z_i (i = 0,1,\ldots,n)$ chosen independently and at random from the interval $(0,1)$, the distribtion of any z_i being the uniform distribution with distribution function given by (1.1). These points are then arranged in increasing order of magnitude as $x_i (i = 0,1,\ldots,n)$. To each point x_i, an interval I is associated where I is given by (1.2). Let X denote the random set which is the point set sum of the $n+1$ intervals $\{I\}$. As was mentioned in Chapter I, this representation is equivalent to the circular representation of Stevens, and, it is this circular nature that will allow us to assume, without loss of generality, that $x_o = 0$. It is also clear that it is only the cyclical or overlapping nature of this representation that differs from the one-dimensional non-circular problem considered by Votaw.

This chapter contains the main result of this thesis; namely that of determining (using the methods of Votaw) the probability function of the measure, $\mu(X)$, of the random set X. Before we derive this function, however, we consider a few related problems.

3.1 Expected Value of the Measure of a Random Linear Set and Related Problems.

First, using methods different from those of Robbins, we again derive equation (2.2.13), i.e. we calculate the expected value of $\mu(X)$. It is clear that $\mu(X)$ may be expressed as

$$\mu(x) = c(x_1) + \sum_{i=1}^{n-1} c(x_{i+1} - x_i) + c(1-x_n) , \tag{3.1.1}$$

where

$$c(x) = \begin{cases} x & 0 \leq x < D , \\ D & D \leq x \leq 1 . \end{cases} \tag{3.1.2}$$

Hence, from (3.1.1), we have the expected value of $\mu(X)$ as

$$E(\mu(X)) = E(c(x_1)) + \sum_{i=1}^{n-1} E(c(x_{i+1} - x_i)) + E(c(1-x_n)) . \tag{3.1.3}$$

Now, $z_1, z_2, \ldots, z_n$ is a random sample from a continuous population (x_0 was fixed at zero), each being uniformly distributed over the interval $(0,1)$. The r th largest of these sample values, namely x_r, is the r th order statistic. It can be shown that the density function $g_r(x_r)$ of x_r is given by

$$(3.1.4) \qquad g_r(x_r) = \frac{n!}{(r-1)!(n-r)!}\left[\int_{-\infty}^{x_r} f(z)dz\right]^{r-1} f(z_r)\left[\int_{x_r}^{\infty} f(z)dz\right]^{n-r},$$

for $-\infty < x_r < \infty$, where $f(z)$ is the common uniform density function over $(0,1)$ of the random variables $z_i (i = 1,2,\ldots,n)$. It can also be proved that the joint density function of x_r for $r < s$ is given by

$$(3.1.5) \qquad g(x_r, x_s) = \frac{n!f(z_r)f(z_s)}{(r-1)!(s-r-1)!(n-s)!}\left[\int_{-\infty}^{x_r} f(z)dz\right]^{r-1}$$

$$\cdot\left[\int_{x_r}^{x_s} f(z)dz\right]^{s-r-1}\left[\int_{x_s}^{\infty} f(z)dz\right]^{n-s},$$

for $-\infty < x_r < x_s < \infty$. It is clear, therefore, that

$$E(c(x_1)) = \int_R c(x_1)g_1(x_1)dR,$$

where R is the region in which the density function $g_1(x_1)$ is defined. From (3.1.2) and (3.1.4) we obtain

$$E(c(x_1)) = n\int_0^D x_1(1-x_1)^{n-1}dx_1 + nD\int_D^1 (1-x_1)^{n-1}dx_1$$

$$(3.1.6) \qquad = \frac{1}{n+1} - \frac{(1-D)^{n+1}}{n+1}.$$

Similarly, it may be shown that

$$(3.1.7) \quad E(c(1-x_n)) = \frac{1}{n+1} - \frac{(1-D)^{n+1}}{n+1} .$$

Now, consider

$$(3.1.8) \quad E(c(x_{i+1}-x_i)) = \iint_{R'} c(x_{i+1}-x_i)\, g(x_{i+1}, x_i)\, dR' ,$$

where, from (3.1.2)

$$(3.1.9) \quad c(x_{i+1}-x_i) = \begin{cases} x_{i+1} - x_i & 0 \le x_{i+1}-x_i < D \\ D & D \le x_{i+1}-x_i < 1 , \end{cases}$$

and, from (3.1.5)

$$(3.1.10) \quad g(x_{i+1}, x_i) = \frac{n!}{(i-1)!(n-i-1)!} x_i^{i-1} (1-x_{i+1})^{n-i-1} \quad 0 < x_i < x_{i+1} < 1 ,$$

and, where R' is the triangular region of the (x_i, x_{i+1}) plane bounded by the lines $x_i = 0$, $x_{i+1} = 1$, and $x_{i+1} = x_i$. In this region, however, the function $c(x_{i+1}-x_i)$ assumes two different values, namely the constant value D in the triangular region, R_1' say, bounded by the lines

$x_i = 0, x_{i+1} = 1$ and $x_{i+1} = x_i + D$, and the value $x_{i+1} - x_i$ in the trapezoidal region, R_2' say, bounded by the lines $x_i = 0$, $x_{i+1} = x_i + D$, $x_{i+1} = D$, and $x_{i+1} = x_i$. Since these regions are disjoint, we may write (3.1.8) as

$$E(c(x_{i+1}-x_i)) = \iint_{R_1'} c(x_{i+1}-x_i)g(x_{i+1},x_i)\,dR_1' + \iint_{R_2'} c(x_{i+1}-x_i)g(x_{i+1},x_i)\,dR_2'.$$

Using (3.1.9) and (3.1.10), this becomes

$$(3.1.11)\quad E(c(x_{i+1}-x_i)) = \frac{n!}{(i-1)!(n-i-1)!}$$

$$\cdot \left\{ D \int_0^{1-D} \int_{x_i+D}^{1} x_i^{i-1}(1-x_{i+1})^{n-i-1}\,dx_{i+1}\,dx_i \right. +$$

$$+ \int_0^{D} \int_0^{x_{i+1}} (x_{i+1}-x_i)\,x_i^{i-1}(1-x_{i+1})^{n-i-1}\,dx_i\,dx_{i+1} +$$

$$\left. + \int_D^{1} \int_{x_{i+1}-D}^{x_{i+1}} (x_{i+1}-x_i)x_i^{i-1}(1-x_{i+1})^{n-i-1}\,dx_i\,dx_{i+1} \right\}.$$

Evaluating the integrals of (3.1.11) and combining with (3.1.6) and (3.1.7) we readily obtain

(3.1.12) $$E(\mu(X)) = 1 - (1-D)^{n+1} .$$

The result given by (3.1.12) suggests the following question: how many points, $n+1$, would be required in order to obtain an expected coverage of, say C ? To answer this question, we need only determine an interger n such that

$$C = 1 - (1-D)^{n+1} .$$

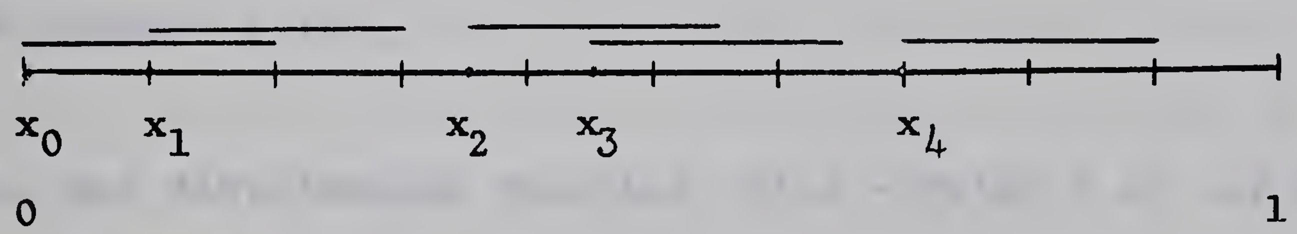
It is clear that

(3.1.13) $$n = \frac{\log (1-C) - \log (1-D)}{\log (1-D)} ,$$

so, the smallest integer greater than $\frac{\log (1-C) - \log (1-D)}{\log (1-D)}$ would give required result. Similarly, for a given n, we can easily determine the size of the interval D that would be needed to yield an expected coverage C.

Another question which leads to a useful result is the following: given $n+1$ points, chosen independently and at random from the unit interval $(0,1)$ (from which we can easily calculate the coverage $\mu(X)$ from (3.1.1)), what would be the expected increase in coverage resulting from the random choice of one more point, u say, from $(0,1)$?

Let ΔC denote the random variable representing this increase in coverage and let it assume the values c. In any such problem $E(\Delta C)$ can be determined as in the following example. Suppose that we are given that $D = 1/5$ and that the points $x_0 = 0.00$, $x_1 = 0.10$, $x_2 = 0.35$, $x_3 = 0.45$, and $x_4 = 0.70$ are the ordered values of 4 points which were chosen independently and at random from $(0,1)$.



First, we calculate the distribution function of ΔC. Now, since there is a finite probability that there is no increase in coverage as a result of choosing the point u, the probability function of ΔC will have a saltus point at $c = 0$ of an amount equal to $Pr(\Delta C = 0) = 0.10 + 0.10 = 0.20$.

Let $0.00 < c < 0.05$, then

$$\begin{aligned} Pr(\Delta C \leq c) = 0.20 &+ Pr(x_1 \leq u \leq x_1 + c) + Pr(x_2 - c \leq u \leq x_2) \\ &+ Pr(x_3 \leq u \leq x_3 + c) + Pr(x_4 - c \leq u \leq x_4) \\ &+ Pr(x_4 \leq u \leq x_4 + c) + Pr(1 - c \leq u \leq 1) \end{aligned}$$

$$= 0.2 + 6c \; .$$

$$Pr(\Delta C = 0.05) = 0.15 + 0.15 = 0.30 \; .$$

Now, let $0.05 < c < 0.10$, then

$$Pr(\Delta C \le c) = 0.80 + Pr(x_4 + 0.05 \le u \le x_4 + c) + Pr(1 - c \le u \le 0.95)$$

$$= 0.70 + 2c \; .$$

$$Pr(\Delta C = 0.10) = 0.10 \; .$$

Hence, the distribution function $F(c) = Pr(\Delta C \le c)$ of the random variable ΔC is

$$F(c) = \begin{cases} 0.20 & c = 0.00 \; , \\ 6c + 0.20 & 0.00 < c < 0.05 \; , \\ 0.80 & c = 0.05 \; , \\ 2c + 0.70 & 0.05 < c < 0.10 \; , \\ 1.00 & c = 0.10 \; . \end{cases}$$

We readily obtain, therefore,

$$E(\Delta C) = (0.20)(0.00) + 6 \int_{0.00}^{0.05} c \, dc + (0.30)(0.05)$$

$$+ 2 \int_{0.05}^{0.10} c \, dc + (0.10)(0.10)$$

$$= 0.04 \; .$$

So, for this particular example, we would expect that the random choice of one more point from the unit interval would increase the coverage by 0.04 .

The previous problem might lead one to pose the following question: given $n+1$ points, which have been chosen independently and at random from the interval $(0,1)$, on the average, how many more points would one have to choose from $(0,1)$ until an increase in coverage is observed ? Let N be a random variable taking values corresponding to the number of points chosen until an increase in coverage is observed. Returning to our previous example, it is clear, therefore, that the random variable N may assume the values $1, 2, 3, \ldots$ with probabilities 0.8 , $(0.2)(0.8)$, $(0.2)^2(0.8)$, $\ldots$ respectively . Hence

$$
\begin{aligned}
E(N) &= \sum_{k=1}^{\infty} k(0.2)^{k-1}(0.8) \\
&= 0.8 \sum_{k=1}^{\infty} k(0.2)^{k-1} \\
&= 0.8/(1.0 - 0.2)^2 = 5/4 \quad .
\end{aligned}
$$

Therefore, we would expect an increase in coverage if two more points are chosen.

3.2 The Probability Distribution of the Measure of a Random Linear Set for a Specified Number (Three) of Points

For the sake of completeness and as a verification of the main result of this thesis, we would now like to evaluate, using methods quite different from those of Votaw, the probability (and distribution) function of the random coverage $\mu(X)$ for the special case $n=2$ (i.e. for 3 points). From (3.1.1) we have

$$\mu(X) = c(x_1) + c(x_2 - x_1) + c(1 - x_n) \quad . \tag{3.2.1}$$

Consider

$$E(e^{\mu(X)t}) = \int\int_R e^{\mu(x)t}\, g(x_1, x_2)\, dR \quad , \tag{3.2.2}$$

where $g(x_1, x_2)$ denotes the joint density function of the order statistics x_1 and x_2, and R denotes the region over which this density is defined. It can be shown that the joint density function of n order statistics $x_1, x_2, \ldots, x_n$ (where the density function of each random variable before the ordering process is the uniform density over $(0,1)$) is $n!$ for $0 < x_1 < x_2 < \ldots < x_n < 1$.

Therefore (3.2.2) becomes

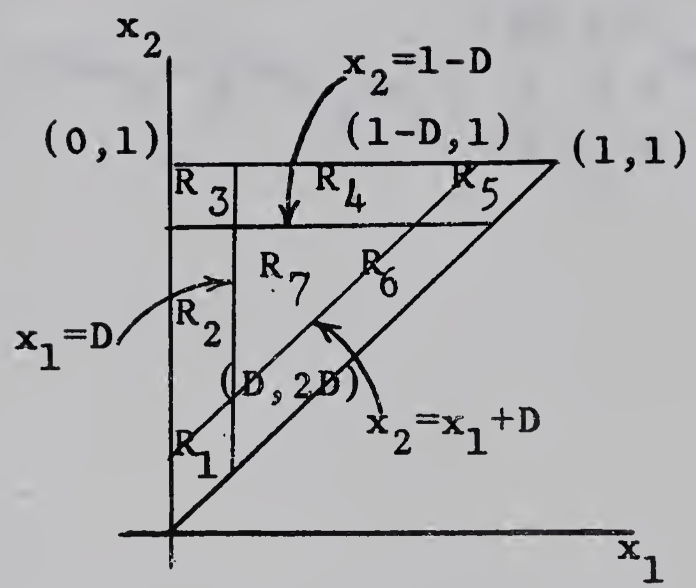
$$E(e^{\mu(X)t}) = 2! \int\int_R e^{(c(x_1)+c(x_2-x_1)+c(1-x_n))t}\, dR \,, \tag{3.2.3}$$

where R , being the region over which the density function

$$g(x_1, x_2) = \begin{cases} 2! & 0 < x_1 < x_2 < 1 \,, \\ 0 & \text{elsewhere} \,, \end{cases}$$

is defined, consists of the triangular region in the (x_1, x_2) plane bounded by the lines $x_1 = 0$, $x_2 = 1$, and $x_1 = x_2$. However, due to the possible overlapping of the intervals, as the values of x_1 and x_2 vary within this triangular region R , the values of the functions defining $\mu(X)$, namely $c(x_1)$, $c(x_2 - x_1)$, and $c(1 - x_2)$ (see (3.1.2)), also change, so that in order to evaluate (3.2.3) , we are forced to integrate over complicated regions within R . (It will be seen shortly how the number and form of these regions depend on the value of D.)

Suppose, now, that $0 < D < 1/3$. The form of the function $c(x)$ dictates that, in order to evaluate (3.2.3) , the region R must be decomposed into seven disjoint regions $R_i (i = 1, 2, \ldots, 7)$, as shown in the figure below.



Notice that if $D = 1/3$, the region R_7 disappears. Now, since R is the union of disjoint regions $R_i (i = 1,2,\ldots,7)$, (3.2.3) becomes

$$(3.2.4) \quad \frac{1}{2} E(e^{\mu(X)t}) = \sum_{i=1}^{7} \int\int_{R_i} e^{(c(x_1) + c(x_2 - x_1) + c(1 - x_2))t} \, dR_i$$

$$= \int_0^D \int_{x_1}^{x_1+D} e^{(x_2 + D)t} dx_2 dx_1 + \int_0^D \int_{x_1+D}^{1-D} e^{(x_1 + 2D)t} dx_2 dx_1$$

$$+ \int_0^D \int_{1-D}^{1} e^{(x_1 + D + 1 - x_2)t} dx_2 dx_1 + \int_{1-D}^{1} \int_D^{x_2-D} e^{(2D+1-x_2)t} dx_1 dx_2$$

$$+ \int_{1-D}^{1} \int_{x_2-D}^{x_2} e^{(D-x_1+1)t} dx_1 dx_2 + \int_{D}^{1-2D} \int_{x_1}^{x_1+D} e^{(2D+x_2-x_1)t} dx_2 dx_1$$

$$+ \int_{1-2D}^{1-D} \int_{x_1}^{1-D} e^{(2D+x_2-x_1)t} dx_2 dx_1 + \int_{D}^{1-2D} \int_{x_1+D}^{1-D} e^{3Dt} dx_2 dx_1 \quad ,$$

from which

$$(3.2.5) \quad E(e^{\mu(X)t}) = 12 \frac{e^{3Dt}}{t^2} - 18 \frac{e^{2Dt}}{t^2} + 6 \frac{e^{Dt}}{t^2} + 6(1-3D) \frac{e^{3Dt}}{t} + 6(2D-1) \frac{e^{2Dt}}{t} + (1-6D+9D^2) e^{3Dt} \quad .$$

It is easy to show that $\lim_{t \to 0} E(e^{\mu(X)t}) = 1$, so that $E(e^{\mu(X)t})$ is indeed a moment generating function of the distribution of $\mu(X)$ (for the case $0 < D < 1/3$) . Since the coefficient of $t^r/r!$ in the expansion of $E(e^{\mu(X)t})$ as a power series is the r th moment of $\mu(X)$, it is easy to show that

$$(3.2.6) \quad E(\mu(X)) = D^3 - 3D^2 + 3D = 1 - (1-D)^3 \quad ,$$

and

$$E(\mu^2(X)) = \frac{17}{2} D^4 - 16D^3 + 9D^2 \quad ,$$

from which we obtain

$$(3.2.7) \qquad \mathrm{Var}\,(\mu(X)) = -\,D^6 + 6D^5 - \frac{13}{2} D^4 + 2D^3 \quad .$$

Consider now the function

$$(3.2.8) \qquad \varphi_1(x) = \begin{cases} K & -\infty < x \le a \quad , \\ 0 & x > a \quad , \end{cases}$$

where K and a are any real constants. Then

$$(3.2.9) \qquad \int_{-\infty}^{\infty} \varphi_1(x)\, e^{xt}\, dx = K\frac{e^{at}}{t} \quad .$$

Similarly, if

$$(3.2.10) \qquad \varphi_2(x) = \begin{cases} aK(1 - \frac{x}{a}) & -\infty < x \le a \quad , \\ 0 & x > a \quad , \end{cases}$$

then

$$(3.2.11) \qquad \int_{-\infty}^{\infty} \varphi_2(x)\, e^{xt}\, dx = K\,\frac{e^{at}}{t^2} \quad .$$

Now, an integral of the form (3.2.9) or (3.2.11) can be associated with each term of the expression (3.2.5) (with the exception of the last term which corresponds to a saltus point in the probability function of $\mu(X)$) . Therefore, (3.2.5) can be written as

$$E(e^{\mu(X)t}) = \int_{-\infty}^{D} 36D(1-\frac{x}{3D})e^{xt}dx - \int_{-\infty}^{2D} 36D(1-\frac{x}{2D})e^{xt}dx + \int_{-\infty}^{D} 6D(1-\frac{x}{D})e^{xt}dx$$

$$+ \int_{-\infty}^{D} 6(1-3D)e^{xt}dx + \int_{-\infty}^{2D} 6(2D-1)e^{xt}dx + (1-6D+9D^2)e^{3Dt}$$

(3.2.12)

$$= 6\int_{D}^{2D} (x-D)e^{xt}dx + 6\int_{2D}^{3D} (-2x+3D+1)e^{xt}dx + (1-6D+9D^2)e^{3Dt} .$$

The term $(1-6D+9D^2)e^{3Dt}$ corresponds to a saltus or jump point in the probability function at the point $x=3D$ with frequency $1-6D+9D^2$. Now, it follows that the probability function $f_2(x)$ of $\mu(X)$ is given by

$$(3.2.13) \quad f_2(x) = \begin{cases} 0 & -\infty < x \le D , \\ 6(x-D) & D \le x \le 2D , \\ 6(-2x+3D+1) & 2D < x < 3D , \\ 1-6D+9D^2 & x = 3D , \\ 0 & x > 3D , \end{cases}$$

provided $0 < D < 1/3$. The function given by (3.2.13) is continuous over the range $D \leq x \leq 3D$ with the exception of $x = 2D$ and $x = 3D$ ($x = 3D$ being a saltus point) . Integrating $f_2(x)$, the distribution function $F_2(x)$ of $\mu(X)$ is found to be

$$(3.2.14) \quad F_2(x) = \begin{cases} 0 & -\infty < x \leq D \,, \\ 3(x-D)^2 & D \leq x \leq 2D \,, \\ -6x^2 + 6(1+3D)x - 9D^2 - 12D & 2D \leq x < 3D \,, \\ 1 & x \geq 3D \,, \end{cases}$$

which is continuous in the range $D \leq x \leq 3D$ except at the saltus point $x = 3D$.

For $D = 1/3$, the probability and distribution functions may be obtained by the same method used above. As mentioned previously, for $D = 1/3$, the region R_7 of the previous figure disappears so that we need only integrate over six regions in the (x_1 , x_2) plane. Equations (3.2.5) , (3.2.6) , (3.2.7), (3.2.13) , and (3.2.14) all hold for the particular case $D = 1/3$. It is interesting to note that in (3.2.13) , the saltus point at $x = 3D = 1$ disappears and the probability function is continuous at this point. (This saltus point was the contribution of the region R_7 to the probability function of $\mu(X)$.) The probability function of $\mu(X)$ for the case $D = 1/3$ is

$$(3.2.15)\quad f_2(x) = \begin{cases} 0 & -\infty < x \leq \frac{1}{3}\ , \\ 2(3x-1) & \frac{1}{3} \leq x \leq \frac{2}{3}\ , \\ 12(1-x) & \frac{2}{3} < x \leq 1\ , \\ 0 & x \geq 1\ . \end{cases}$$

This function is continuous over the range $D \leq x \leq 1$ except at the point $x = 2D$. Integrating (3.2.15) , we obtain

$$(3.2.16)\quad F_2(x) = \begin{cases} 0 & -\infty < x \leq \frac{1}{3}\ , \\ \frac{1}{3}\,(3x-1)^2 & \frac{1}{3} \leq x \leq \frac{2}{3}\ , \\ -6x^2 + 12x - 5 & \frac{2}{3} \leq x \leq 1\ , \\ 1 & x \geq 1\ , \end{cases}$$

which is continuous in the range $D \leq x \leq 1$.

In a similar way, we can derive the corresponding results for other values of D $(0 < D < 1)$. These results are summarized below.

For the case $1/3 \leq D \leq 1/2$ we have

$$(3.2.17)\qquad E(\mu(X)) = 1 - (1-D)^3$$

$$(3.2.18)\qquad \mathrm{Var}\,(\mu(X)) = -D^6 + 6D^5 - \frac{13}{2}D^4 + 2D^3\ ,$$

$$(3.2.19) \quad f_2(x) = \begin{cases} 0 & -\infty < x \leq D , \\ 6(x-D) & D \leq x \leq 2D , \\ 6(-2x+3D+1) & 2D < x < 1 , \\ 1-6D+9D^2 & x = 1 , \\ 0 & x > 1 , \end{cases}$$

which is continuous in the range $D \leq x \leq 1$ except at the points $x = 2D$ and $x = 1$ (this point being a saltus of amount $1-6D+9D^2$), and

$$(3.2.20) \quad F_2(x) = \begin{cases} 0 & -\infty < x \leq D , \\ 3(x-D)^2 & D \leq x \leq 2D , \\ -6x^2+6(1+3D)x-9D^2-12D & 2D \leq x < 1 , \\ 1 & x \geq 1 , \end{cases}$$

which is continuous except at the saltus point $x = 1$.

The case $1/2 \leq D < 1$ yields

$$(3.2.21) \qquad E(\mu(X)) = 1 - (1-D)^3 ,$$

$$(3.2.22) \qquad \mathrm{Var}\,(\mu(X)) = -D^6+6D^5-\frac{29}{2}D^4-12D^2+4D-\frac{1}{2} ,$$

$$(3.2.23)\quad f_2(x) = \begin{cases} 0 & -\infty < x \leq D \;, \\ 6(x-D) & D \leq x < 1 \;, \\ -2+6D-3D^2 & x = 1 \;, \\ 0 & x > 1 \;, \end{cases}$$

which is continuous in the range $D \leq x \leq 1$ with the exception of a saltus of amount $-2 + 6D - 3D^2$ at $x = 1$, and

$$(3.2.24)\quad F_2(x) = \begin{cases} 0 & -\infty < x \leq D \;, \\ 3(x-D)^2 & D \leq x < 1 \;, \\ 1 & x \geq 1 \;, \end{cases}$$

which, is continuous in the range $D \leq x \leq 1$ with the exception of the saltus at $x=1$.

3.3 The Probability Distribution of the Measure of a Random Linear Set for an Unspecified Number of Points .

We now derive the main result of this thesis. We show, using the methods of Votaw, that the probability function $f_n(x)$ of this random variable $\mu(X)$ (which was defined at the beginning of this and the first chapters) is given by (1.3) .

Recall that our circular representation of the problem was the following. Order $n+1$ points chosen independently and at random on a circle of unit circumference, as $x_i (i = 0,1,\ldots,n)$. To each point, attach an arc of length $D(0 < D < 1)$; the point being at the clockwise end of the arc. Choose the beginning of any one arc, x_o say, as the origin and measure distances in an anti-clockwise direction around the circumference. Let $\mu(X)$ be that portion of the circumference which is covered by the $n+1$ arcs. Make the transformation

$$(3.3.1) \quad y_o = x_o = 0 \; , \; y_i = x_i - x_{i-1} (i = 1,2,\ldots,n) \; , \; y_{n+1} = 1 - x_n \; .$$

It is clear that $y_i \geq 0 \; (i = 1,2,\ldots,n+1)$ and $\sum_{i=1}^{n+1} y_i = 1$,

so that we will be concerned only in those points $(y_1, y_2, \ldots, y_{n+1})$ which lie on that portion of the hyperplane $y_1 + y_2 + \ldots + y_{n+1} = 1$ for which $y_1 \geq 0, y_2 \geq 0, \ldots, y_{n+1} \geq 0$.

Denote this region by S . The content of S , $V(S)$ say, is given by Hall's result (see (2.3.4)) as

$$V(S) = \frac{\sqrt{n+1}}{n!} \quad . \tag{3.3.2}$$

If R is any region of S, with content $V(R)$, then since points $(y_1, y_2, \ldots, y_{n+1})$ are equally likely to be at any position in S, the probability that a point p belongs to R is

$$\Pr(p \in R) = \frac{V(R)}{V(S)} = \frac{n!}{\sqrt{n+1}} V(R) \quad . \tag{3.3.3}$$

(Note that from (3.3.2) it follows that the joint density function of $y_1, y_2, \ldots, y_{n+1}$ $(y_i \geq 0 ; i=1,2,\ldots,n+1; \sum_{i=1}^{n+1} y_i = 1)$ is just $n!/\sqrt{n+1}$.)

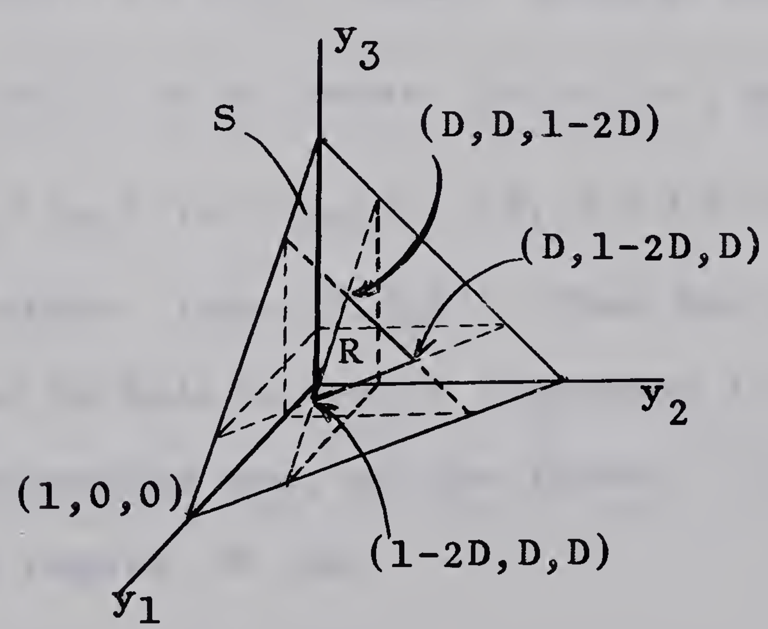
Now, we have seen that $\mu(X)$ is given by

$$\mu(X) = c(x_1) + \sum_{i=1}^{n-1} c(x_{i+1} - x_i) + c(1-x_n) \quad ,$$

where the function $c(x)$ was defined by (3.1.2) . The range of $\mu(X)$ is $D \leq \mu(X) \leq m$, where m denotes the minimum of 1 and $(n+1)D$. If $m = (n+1)D$, there is a finite probability that the set X will consist of $n+1$ disjoint arcs of length D , and hence the distribution of

$\mu(X)$ will have a saltus point at $\mu(X) = (n+1)D$. Given that $m = (n+1)D$, $\mu(X) = (n+1)D$ if and only if $y_1 \geq D, y_2 \geq D, \ldots, y_{n+1} \geq D$. Let R be the subregion of S consisting of all points $(y_1, y_2, \ldots, y_{n+1})$ which belong to S but also satisfy the condition that $y_1 \geq D, y_2 \geq D, \ldots, y_{n+1} \geq D$. It is clear from the diagram below that R will be the interior of the convex polyhedron whose vertices are

$$V_1 \ (1-nD, D, \ldots, D)$$
$$V_2 \ (D, 1-nD, \ldots, D)$$
$$\vdots$$
$$V_{n+1} (D, D, \ldots, 1-nD) .$$



We now apply Hall's result (namely equation (2.3.4)) to evaluate the content of the region R. The content of this region will be unaffected by translation by the vector $(-D,-D,\ldots,-D)$ into the polyhedron with vertices

$$\begin{array}{ll} V_1' & (1-(n+1)D, 0, \ldots, 0) \\ V_2' & (0, 1-(n+1)D, \ldots, 0) \\ \vdots & \\ V_{n+1}' & (0, 0, \ldots, 1-(n+1)D) \end{array} .$$

But these are simply the points of intersection with the coordinate axes of the hyperplane $z_1 + z_2 + \ldots + z_{n+1} = 1-(n+1)D$ $(z_i = y_i - D\,;\ i = 1,2,\ldots,n+1)$. Now, in order to apply Hall's result, we must apply a uniform stretch. The substitution $q_i = z_i/(1-(n+1)D)$ $(i = 1,2,\ldots n+1)$ expands every dimension $z_i (i = 1,2,\ldots,n+1)$ by a factor $1-(n+1)D$, giving the hyperplane $q_1 + q_2 + \ldots + q_{n+1} = 1$ $(q_i \geq 0\,;\ i = 1,2,\ldots,n+1)$. We have seen before (see (3.3.2)) that the content of this region is given by Hall's result (equation (2.3.4)) as $\sqrt{n+1}/n!$. Contracting now, by the factor $1-(n+1)D$, the content of the region R is

$$\frac{\sqrt{n+1}}{n!}\,(1-(n+1)D)^n .$$

Hence, from (3.3.3), given $m = (n+1)D$,

$$\Pr(\mu(X) = (n+1)D) = \Pr(y_1 \geq D, y_2 \geq D, \ldots, y_{n+1} \geq D)$$

$$= (1-(n+1)D)^n \quad . \tag{3.3.4}$$

Hence, at the point $\mu(X) = (n+1)D$, the probability function of $\mu(X)$ will not be defined and the distribution function will have a saltus of amount $(1-(n+1)D)^n$. Now consider the distribution of $\mu(X)$ on the range $D < \mu(X) < m$ ($\mu(X)$ takes the value D with probability zero).

Let the random variable $\mu(X)$ take the values x and proceed in a manner analogous to the methods of Votaw. The probability that $X < x < X + \Delta X$ (where $X < m$ and ΔX denotes an arbitrary small positive increment in X) can be evaluated by determining volumes of certain regions contained in S (the n-dimensional region consisting of all points $(y_1, y_2, \ldots, y_{n+1})$ which lie in that portion of the hyperplane $y_1 + y_2 + \ldots + y_{n+1} = 1$ satisfying $y_i \geq 0$ $(i = 1, 2, \ldots, n+1)$). Note that since $x < m$, we can always choose ΔX such that $X + \Delta X < m$. Consider the following conditions

(a) $qD \leq x < (q+1)D$ ($q=1,2,\ldots,M$; M denotes the minimum of n and the greatest integer $< \frac{1}{D}$; $x < m$).

For any particular value of q, decompose the set $(y_1, y_2, \ldots, y_j)$ and $(y_j, y_{j+1}, \ldots, y_{n+1})$ such that

(b) $\quad y_u \geq D \qquad (u = 1, 2, \ldots, j ; j \leq q)$,

(c) $\quad y_v < D \qquad (v = j+1, j+2, \ldots, n+1)$.

Now, we have $x = c(x_1) + \sum_{i=1}^{n-1} c(x_{i+1} - x_i) + c(1-x_n) = \sum_{i=1}^{n+1} c(y_i)$,

which, in view of (b) and (c) becomes

$$x = jD + \sum_{v=j+1}^{n+1} y_v \quad . \tag{3.3.5}$$

Now, we have $\sum_{i=1}^{n+1} y_i = \sum_{u=1}^{j} y_u + \sum_{v=j+1}^{n+1} y_v = 1$, which, with (3.3.6) becomes

(d) $\quad \sum_{u=1}^{j} y_u = 1 - x + jD$.

Let $A_{j-1}(x)$ denote the content of the $(j-1)$-dimensional region consisting of all points $(y_1, y_2, \ldots, y_j)$ which satisfy conditions (b) and (d) , and let $B_{n-j}(x)$

be the content of the $(n-j)$-dimensional region which is the intersection of the hyperplane $\sum_{v=j+1}^{n+1} y_v = x - jD$ with the $(n-j+1)$-dimensional hypercube given by condition (c) .

The derivation of $A_{j-1}(x)$ is the same as that of (3.3.4) . $A_{j-1}(x)$ is the volume of the region consisting of all points $(y_1, y_2, \ldots, y_j)$ such that $y_1 \geq D, y_2 \geq D, \ldots, y_j \geq D$ and which lie on the $(j-1)$-dimensional hyperplane $y_1 + y_2 + \ldots + y_j = 1 - x + jD$. We require then, the content of the $(j-1)$-dimensional region with vertices (each having j components)

$$V_1(1-x+D, D, \ldots, D)$$

$$V_2(D, 1-x+D, \ldots, D)$$

$$\vdots$$

$$V_j(D, D, \ldots, 1-x+D) \;.$$

Proceeding as before, translation of this region by the vector $(-D, -D, \ldots, -D)$ will give a region with vertices

$$V_1'(1-x,0,\ldots,0)$$
$$V_2'(0,1-x,\ldots,0)$$
$$\vdots$$
$$V_j'(0,0,\ldots,1-x) \quad ,$$

which is the intersection of the hyperplane $w_1+w_2+\ldots+w_j=1-x$ $(w_i=y_i-D;\ i=1,2,\ldots,j)$ with the j-dimensional hypercube of side $1-x$. In order to apply Hall's result, we let $q_i=w_i/(1-x)(i=1,2,\ldots,j)$ which expands this region to a region with vertices

$$V_1''(1,0,\ldots,0)$$
$$V_2''(0,1,\ldots,0)$$
$$\vdots$$
$$V_j''(0,0,\ldots,1) \quad ,$$

which is the intersection of the hyperplane $q_1+q_2+\cdots+q_j=1$ with the unit hypercube. Using Hall's result, it is clear then, that the content of this $(j-1)$-dimensional region is $\sqrt{j}/(j-1)!$. Thus, contracting by the same factor, the required content is

$$A_{j-1}(x) = \frac{\sqrt{j}}{(j-1)!}(1-x)^{j-1} \quad . \tag{3.3.6}$$

$B_{n-j}(x)$ is the content of the $(n-j)$-dimensional region cut from the hyperplane $y_{j+1} + y_{j+2} + \cdots + y_{n+1} = x - jD$ by the hypercube of dimension $n-j+1$ defined by $y_{j+1} < D, y_{j+2} < D, \ldots, y_{n+1} < D$. In order to apply Hall's result, we expand the above hypercube of side D to the unit hypercube by letting $z_i = y_i/D \; (i = j+1,\ldots,n+1)$, so that, using Hall, we can determine the volume of the $(n-j)$-dimensional region which is the intersection of the hyperplane $z_{j+1} + z_{j+2} + \cdots + z_{n+1} = x/D - j$ with the unit hypercube $z_{j+1} < 1, \ldots, z_{n+1} < 1$. Now since

$$qD \leq x < (q+1)D$$

may be written as

$$q-j \leq \frac{x}{D} - j < (q+1) - j \quad ,$$

we have $[x/D - j] = q - j$, so that, from (2.3.4), the content of the expanded region is

$$\frac{\sqrt{n-j+1}}{(n-j)!} \sum_{r=0}^{q-j} (-1)^r \binom{n-j+1}{r} \left(\frac{x}{D} - j - r\right)^{n-j} .$$

On making the appropriate contraction, we obtain

$$(3.3.7) \qquad B_{n-j}(x) = \frac{\sqrt{n-j+1}}{(n-j)!} \sum_{r=0}^{q-j} (-1)^r \binom{n-j+1}{r} [x-D(j+r)]^{n-j} .$$

Now, $A_{j-1}(x)$ and $B_{n-j}(x)$ define contents of regions contained in orthogonal subspaces of the n-dimensional hyperplane $y_1 + y_2 + \dots + y_{n+1} = 1$ (i.e. the region S). $A_{j-1}(x)$ is the content of a $(j-1)$-dimensional region which belongs to a subspace of a vector space consisting of $(n+1)$-tuples $(y_1, y_2, \dots, y_j, 0, \dots, 0)$ while $B_{n-j}(x)$ is equal to the content of an $(n-j)$-dimensional region which belongs to a subspace of a vector space consisting of $(n+1)$-tuples $(0, \dots, 0, y_{j+1}, \dots, y_{n+1})$. These vector spaces are orthogonal compliments whose product space is the positive orthant of $(n+1)$-dimensional Euclidean space. The n-dimensional region S belongs to this product space and we wish to determine the content of that subregion of S , $T(x)$ say, which is the resulting product of the regions with contents given by $A_{j-1}(x)$ and $B_{n-j}(x)$. That is, $T(x)$ is the content of the $(j-1)+(n-j) = (n-1)$ - dimensional subregion of the n-dimensional hyperplane S which consists of all points $(y_1, y_2, \dots, y_{n+1})$ which satisfy conditions (b) , (c), and (d) (it is clear that, for points

belonging to S , one of condition (d) or equation (3.3.5) is redundant). The content of $T(x)$ is therefore, $A_{j-1}(x)\, B_{n-j}(x)$, which, from (3.3.6) and (3.3.7) , is given by

$$(3.3.8) \quad T(x) = \frac{\sqrt{j}}{(j-1)!} (1-x)^{j-1} \frac{\sqrt{n-j+1}}{(n-j)!} \sum_{r=0}^{q-j} (-1)^r \binom{n-j+1}{r} \cdot [x-D(j+r)]^{n-j} \quad .$$

Let us restate the above conditions (i.e. (b) , (c) , (d) , and (3.3.5)) as

(i) $\quad y_1 \geq D, \ldots, y_j \geq D$,

(ii) $\quad y_{j+1} < D, \ldots, y_{n+1} < D$,

(iii) $\quad y_1 + y_2 + \ldots + y_j = 1 - x + jD$,

(iv) $\quad y_{j+1} + y_{j+2} + \ldots + y_{n+1} = x - jD$.

We wish to determine $\Pr(X < x < X + \Delta X)$, given that these conditions are satisfied. Now, since (iii) and $\sum_{i=1}^{n+1} y_i = 1$ imply (iv) , $T(x)$ is completely determined as that region of S satisfying conditions (i) , (ii) , and (iv) . In other words, consider the subregion S^* of S for which

(i) and (ii) are true. Then $T(x)$ is the intersection of this n-dimensional region S^* with the hyperplane, $J(x)$ say, with equation given by (iv) (it is clear that we could also have chosen $J(x)$ to be the hyperplane given by (iii)). Let us consider an example. Suppose we choose 3 points on the circle of unit circumference, so that $n+1=3$. Furthermore, suppose $j=2$. Then the above conditions become

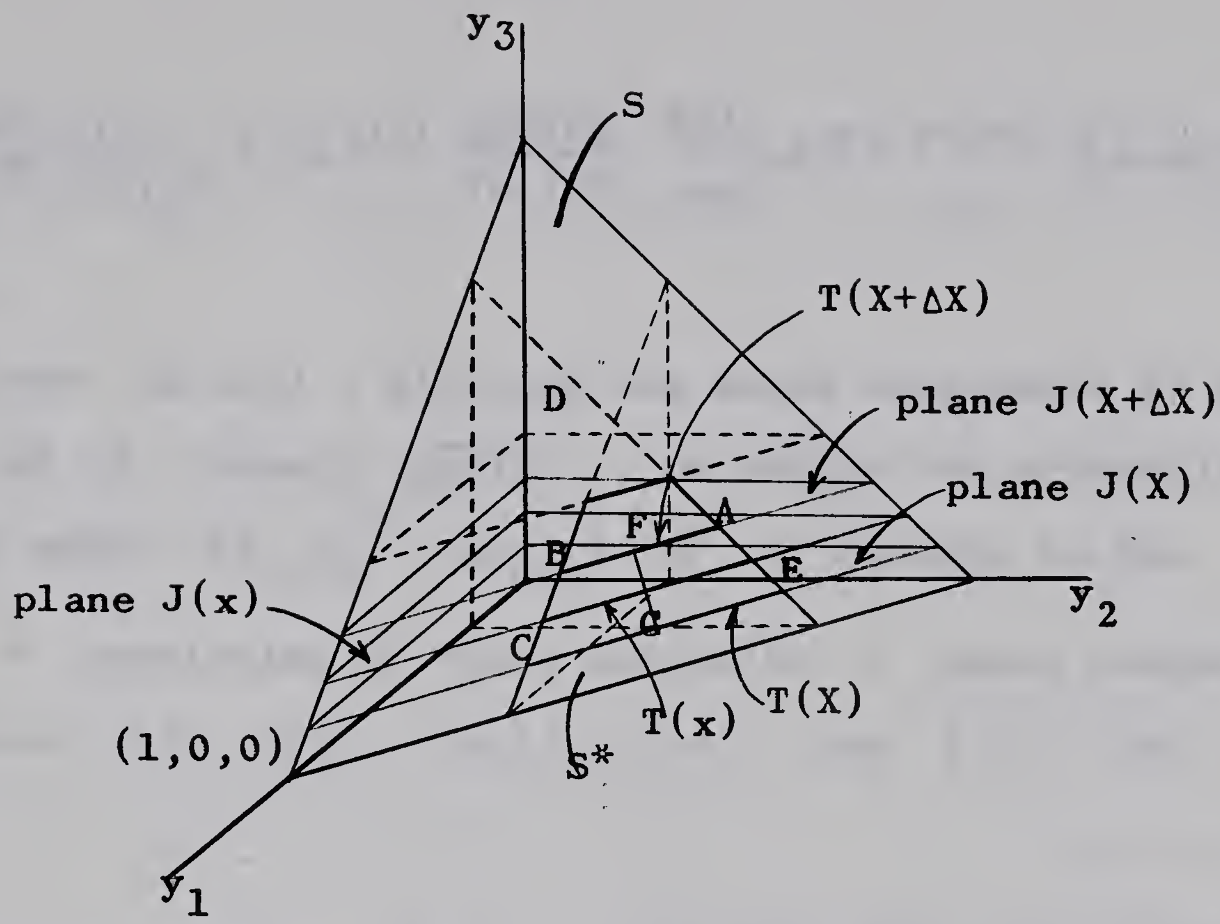
(i) $\quad y_1 \geq D, y_2 \geq D$,

(ii) $\quad y_3 < D$,

(iii) $\quad y_1 + y_2 = 1 - x + 2D$,

(iv) $\quad y_3 = x - 2D$.

Now, since we want to determine $\Pr(X < x < X + \Delta X)$, we must have $X < x < X + \Delta X$, so, not only must y_3 satisfy condition (ii) , but it must also lie in the region $X - 2D < y_3 < X + \Delta X - 2D$, that is, between the planes $J(X)$ and $J(X + \Delta X)$.



Hence, in order to determine the probability that $X < x < X + \Delta X$ given that conditions (i), (ii), (iii), and (iv) are satisfied, it is clear from the figure above that we need only evaluate the content of the region ABCE. For small ΔX, this content is given by the length of FG (which is just the distance between the planes $J(X)$ and $J(X+\Delta X)$ measured along the face of the tetrahedron) multiplied by $T(x)$ (the content of the $(n-1=1)$-diminsional region which is the intersection of S^* with the plane $J(x)$). It can be shown that, in the general case, the length of FG is $\dfrac{\Delta X\sqrt{n+1}}{\sqrt{n-j+1}\,\sqrt{j}}$. Therefore, the content of the n-dimensional region consisting of all points $(y_1, y_2, \cdots, y_{n+1})$ satisfying conditions (i), (ii), (iii), and (iv) and the condition $X < x < X + \Delta X$ is, using (3.3.8), given by

$$\frac{\Delta X \sqrt{n+1}\ \sqrt{j}}{\sqrt{n-j+1}\ \sqrt{j}\ (j-1)!}\ (1-x)^{j-1}\ \frac{\sqrt{n-j+1}}{(n-j)!}\ \sum_{r=0}^{q-j} (-1)^r \binom{n-j+1}{r} \left[x-D(j+r)\right]^{n-j} .$$

Hence, from (3.3.3) , dividing the above expression by the content of S , namely $\sqrt{n+1}/n!$, we obtain the probability that any point $(y_1, y_2, \cdots, y_{n+1})$ of S belongs to the region R consisting of those points of S which satisfy conditions (i) , (ii) , (iii) , (iv) , and $X < x < \Delta X$ as

$$\Pr(X < x < X + \Delta X) = \frac{\Delta X\ n!}{(j-1)!(n-j)!}(1-x)^{j-1} \sum_{r=0}^{q-j} (-1)^r \binom{n-j+1}{r} \cdot \left[x-D(j+r)\right]^{n-j} .$$

Now, if any other of the $\binom{n+1}{j}$ combinations of the j y's had been specified in (b) and the n-j+1 complementary y's had been specified in (c) , the corresponding values of $A_{j-1}(x)$ and $B_{n-j}(x)$ would be equal to those in (3.3.8) , so that

$$\Pr(X < x < X + \Delta X) = \Delta X\ n! \sum_{j=1}^{q} \sum_{r=0}^{q-j} \frac{(-1)^r}{(n-j)!(j-1)!} \cdot \binom{n+1}{j} \binom{n-j+1}{r} (1-x)^{j-1} \left[x-D(j+r)\right]^{n-j} .$$

Hence, dividing each side of the above equation by ΔX and letting ΔX tend to zero, we obtain the probability function of $\mu(X)$ as

$$f_n(x) = n! \sum_{j=1}^{q} \sum_{r=0}^{q-j} \frac{(-1)^r}{(n-j)!(j-1)!} \binom{n+1}{j} \binom{n-j+1}{r} (1-x)^{j-1} \cdot \left[x-D(j+r)\right]^{n-j},$$

for $qD \leq x < (q+1)D$ ($q=1,2,\ldots,M$; M denotes the minimum of n and the greatest integer less than $1/D$; $x < m$) .

As we have seen before, $f_n(x)$ is not defined at $x=(n+1)D$ if $(n+1)D < 1$ (see (3.3.4)) and the distribution function of $\mu(X)$ is continuous with exception, in the case $(n+1)D < 1$, of a saltus of amount $(1-(n+1)D)^n$ at $x=(n+1)D$.

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